Inequalities involving upper bounds for certain matrix operators

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Abstract. In this paper, we considered the problem of finding the upper bound Hausdorff matrix operator from sequence spaces $l_p(v)$ (or $d(v, p)$) into $l_p(w)$ (or $d(w, p)$). Also we considered the upper bound problem for matrix operators from $d(v, 1)$ into $d(w, 1)$, and matrix operators from $e(w, \infty)$ into $e(v, \infty)$, and deduce upper bound for Cesaro, Copson and Hilbert matrix operators, which are recently considered in [5] and [6] and similar to that in [10].

Keywords. Inequality; norm; summability matrix; Hausdorff matrix; Hilbert matrix; weighted sequence space; Lorentz sequence space.

1. Introduction

We study the norm of a certain matrix operator on $l_p(w)$ and Lorentz sequence spaces $d(w, p)$, $p \geq 1$, which is considered in [2] on l_p spaces and in [6,7,8] and [9] on $l_p(w)$ and $d(w, p)$ for some matrix operator such as Cesaro, Copson and Hilbert operators.

Let l_p be the normed linear space of all sequences $x = (x_n)$ with finite norm $\|x\|_p$, where

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Let $w = (w_n)$ be a sequence with positive entries. For $p \geq 1$, we define the weighted sequence space $l_p(w)$ as follows:

$$l_p(w) = \left\{ (x_n) \colon \sum_{n=1}^{\infty} w_n |x_n|^p < \infty \right\},$$

with norm, $\| \cdot \|_{p,w}$, which is defined as follows:

$$\|x\|_{w,p} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p \right)^{1/p}.$$

Also, if $w = (w_n)$ is a decreasing sequence of positive number such that $\lim_{n \to \infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$, then the Lorentz sequence space $d(w, p)$ is defined as follows:

$$d(w, p) = \left\{ (x_n) \colon \sum_{n=1}^{\infty} w_n x_n^{*p} < \infty \right\},$$

where (x_n^*) is the decreasing rearrangement of $(|x_n|)$. In fact $d(w,p)$ is the space of null sequences x for which x^* is in $l_p(w)$, with norm $\|x\|_{d(w,p)} = \|x^*\|_{w,p}$.

Let $X_k^* = x_1^* + \cdots + x_k^*$ and $W_k = w_1 + \cdots + w_k$. We define the weighted sequence space $e(w,\infty)$ as follows:

$$e(w,\infty) = \left\{ (x_n) : \sup_k \frac{X_k^*}{W_k} < \infty \right\},$$

with norm $\| \cdot \|_{w,\infty}$, which is defined as follows:

$$\|x\|_{w,\infty} = \sup_k \frac{X_k^*}{W_k}.$$

Our objective in §2 is to give a generalization of some results obtained by Bennett [1,2] and Jameson and Lashkaripour [6], for Hausdorff matrix operators on the weighted sequence space. In §3 we try to solve the problem of finding the norm of matrix operators from $d(v,1)$ into $d(w,1)$, and matrix operators from $e(w,\infty)$ into $e(v,\infty)$, and we deduce upper bound for certain matrix operators such as Cesaro, Copson and Hilbert operators.

2. Hausdorff matrix operator on $l_p(w)$ and $d(w,p)$

In this section, we consider the Hausdorff matrix operator $H(\mu) = (h_{j,k})$, such that

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \triangle^{j-k} a_k, & \text{if } 1 \leq k \leq j, \\ 0, & \text{if } k > j, \end{cases}$$

where $\triangle$ is the difference operator; that is

$$\triangle a_k = a_k - a_{k+1},$$

and (a_k) is a sequence of real numbers, normalized so that $a_1 = 1$.
    If

$$a_k = \int_0^1 \theta^{k-1} \mathrm{d}\mu(\theta), \quad k = 1,2,\ldots,$$

where μ is a probability measure on $[0,1]$, then for all $j,k = 1,2,\ldots$, we have

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \int_0^1 \theta^{k-1}(1-\theta)^{j-k} \mathrm{d}\mu(\theta), & \text{if } 1 \leq k \leq j \\ 0, & \text{if } k > j \end{cases}.$$

The Hausdorff matrix contains the famous classes of matrices. These classes are as follows:

  (i) Choice $\mathrm{d}\mu(\theta) = \alpha(1-\theta)^{\alpha-1}\mathrm{d}\theta$ gives the Cesaro matrix of order α.
 (ii) Choice $\mathrm{d}\mu(\theta) = $ point evaluation at $\theta = \alpha$ gives the Euler matrix of order α.
(iii) Choice $\mathrm{d}\mu(\theta) = \frac{|\log\theta|^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\theta$ gives the Hölder matrix of order α.
 (iv) Choice $\mathrm{d}\mu(\theta) = \alpha\theta^{\alpha-1}\mathrm{d}\theta$ gives the Gamma matrix of order α.

The Cesaro, Hölder and Gamma matrices have non-negative entries, whenever $\alpha > 0$. Also the Euler matrix is non-negative, when $0 \leq \alpha \leq 1$. So, if we obtain the norm of the Hausdorff matrix, then it is also an upper bound for the above matrices.

Now consider the operator A defined by $Ax = y$, where $y_i = \sum_{i=1}^{\infty} a_{i,j} x_j$. We write $\|A\|_{v,w,p}$ for the norm of A as an operator from $l_p(v)$ into $l_p(w)$, and $\|A\|_{w,p}$ for the norm of A as an operator from $l_p(w)$ into itself, and $\|A\|_p$ for the norm of A as an operator from l_p into itself, and $\|A\|_{d(w,p)}$ for the norm of A as an operator from $d(w,p)$ into itself.

The following conditions are needed to convert statements for $l_p(w)$ to ones for $d(w,p)$. We assume throughout that

(1) For all $i, j, a_{i,j} \geq 0$.
(2) For all subsets M, N of natural numbers having m, n elements respectively, we have

$$\sum_{i \in M} \sum_{j \in N} a_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}.$$

(3) $\sum_{i=1}^{\infty} w_i a_{i,1}$ is convergent.
    Condition (1) implies that $|A(x)| \leq A(|x|)$ and hence the non-negative sequences are sufficient to determine norm of A.

PROPOSITION 2.1.

Let $p \geq 1$ and $A = (a_{i,j})$ be an operator with conditions (1) and (2). Then

$$\|A(x)\|_{d(w,p)} \leq \|A(x^*)\|_{d(w,p)},$$

for all non-negative elements x of $d(w,p)$. Hence decreasing, non-negative elements are sufficient to determine norm of A.

Condition (3) ensured that at least finite sequence are mapped into $d(w,1)$.

PROPOSITION 2.2. (Lemma 1 of [5])

Let $p \geq 1$ and $A = (a_{i,j})$ be an operator with non-negative entries. Also, let A map $d(w,p)$ into itself. If for $x \in d(w,p)$, we set $Ax = y$ such that $y_i = \sum_{j=1}^{\infty} a_{i,j} x_j$, then the following conditions are equivalent:

(a) $y_1 \geq y_2 \geq \cdots \geq 0$ when $x_1 \geq x_2 \geq \cdots \geq 0$.
(b) $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$ decreases with i for each n.

In the following statement, we assume (v_n) and (w_n) to be non-negative decreasing sequences with $v_1 = 1$.

Theorem 2.1. *Let $H(\mu)$ be the Hausdorff matrix operator and $p > 1$. Then the Hausdorff matrix operator maps $l_p(v)$ into $l_p(w)$, and*

$$\left(\inf \frac{w_n}{v_n} \right)^{1/p} \int_0^1 \theta^{-1/p} d\mu(\theta)$$

$$\leq \|H\|_{v,w,p} \leq \left(\sup \frac{w_n}{v_n} \right)^{1/p} \int_0^1 \theta^{-1/p} d\mu(\theta).$$

Therefore the Hausdorff matrix operator maps $l_p(w)$ into itself, and

$$\|H\|_{w,p} = \int_0^1 \theta^{-1/p} \mathrm{d}\mu(\theta).$$

Proof. Let x be a non-negative sequence. Since (w_n) is decreasing, and applying Theorem 216 of [3], we have

$$\|Hx\|_{w,p}^p = \sum_{j=1}^{\infty} w_j \left(\sum_{k=1}^{j} \binom{j-1}{k-1} \left(\int_0^1 \theta^{k-1}(1-\theta)^{j-k} \mathrm{d}\mu(\theta) \right) x_k \right)^p$$

$$\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} \binom{j-1}{k-1} \left(\int_0^1 \theta^{k-1}(1-\theta)^{j-k} \mathrm{d}\mu(\theta) \right) w_k^{1/p} x_k \right)^p$$

$$\leq \left(\int_0^1 \theta^{-1/p} \mathrm{d}\mu(\theta) \right)^p \sum_{j=1}^{\infty} w_j x_j^p$$

$$= \left(\int_0^1 \theta^{-1/p} \mathrm{d}\mu(\theta) \right)^p \sum_{j=1}^{\infty} \frac{w_j}{v_j} v_j x_j^p$$

$$\leq \sup \frac{w_j}{v_j} \left(\int_0^1 \theta^{-1/p} \mathrm{d}\mu(\theta) \right)^p \|x\|_{v,p}^p.$$

Hence

$$\|Hx\|_{w,p} \leq \left(\sup \frac{w_n}{v_n} \right)^{1/p} \left(\int_0^1 \theta^{-1/p} \mathrm{d}\mu(\theta) \right) \|x\|_{v,p},$$

and so

$$\|H\|_{v,w,p} \leq \left(\sup \frac{w_n}{v_n} \right)^{1/p} \int_0^1 \theta^{-1/p} \mathrm{d}\mu(\theta).$$

It remains to prove the left-hand inequality. We take

$$0 < \delta < \frac{1}{p}, \quad x_n = (n)^{-\frac{1}{p} - \delta}$$

and any positive ε, where $0 < \varepsilon < 1$; and choose α, N, and δ such that

$$\left(1 + \frac{1}{\alpha} \right)^{-2/p} > 1 - \varepsilon,$$

$$\int_{\alpha/n}^1 \theta^{-1/p} \mathrm{d}\mu(\theta) > (1 - \varepsilon) \int_0^1 \theta^{-1/p} \mathrm{d}\mu(\theta), \quad n \geq N,$$

$$\sum_{n=N}^{\infty} w_n x_n^p > (1 - \varepsilon) \sum_{n=1}^{\infty} w_n x_n^p.$$

Since $(x_n) \in l_p$, and $0 < v_n \leq 1$, we deduce that $(x_n) \in l_p(v)$. Also, we have

$$(Hx)_n = \sum_{m=1}^{n} \binom{n-1}{m-1} \left(\int_0^1 \theta^{m-1}(1-\theta)^{n-m} d\mu(\theta) \right) x_m$$

$$\geq (1-\varepsilon)^2 x_n \int_0^1 \theta^{-1/p} d\mu(\theta), \quad n \geq N,$$

and so

$$w_n^{1/p}(Hx)_n \geq (1-\varepsilon)^2 w_n^{1/p} x_n \int_0^1 \theta^{-1/p} d\mu(\theta), \quad n \geq N.$$

Hence

$$\|Hx\|_{w,p}^p \geq \sum_{n=N}^{\infty} w_n (Hx)_n^p$$

$$\geq (1-\varepsilon)^{2p} \left(\int_0^1 \theta^{-1/p} d\mu(\theta) \right)^p \sum_{n=N}^{\infty} w_n x_n^p$$

$$\geq (1-\varepsilon)^{2p+1} \left(\int_0^1 \theta^{-1/p} d\mu(\theta) \right)^p \sum_{n=1}^{\infty} w_n x_n^p$$

$$= (1-\varepsilon)^{2p+1} \left(\int_0^1 \theta^{-1/p} d\mu(\theta) \right)^p \sum_{n=1}^{\infty} \frac{w_n}{v_n} v_n x_n^p$$

$$\geq \inf \frac{w_n}{v_n} (1-\varepsilon)^{2p+1} \left(\int_0^1 \theta^{-1/p} d\mu(\theta) \right)^p \|x\|_{v,p}^p.$$

Since ε is arbitrary, if $\varepsilon \longrightarrow 0$, we have

$$\|Hx\|_{w,p}^p \geq \inf \frac{w_n}{v_n} \left(\int_0^1 \theta^{-1/p} d\mu(\theta) \right)^p \|x\|_{p,v},$$

and this completes the proof of the theorem. $\square$

COROLLARY 2.1.

Let $p > 1$ and $p^* = \frac{p}{p-1}$. Then Cesaro, Hölder, Gamma and Euler operators map $l_p(w)$ into $l_p(w)$. Also, we have

$$\|C(\alpha)\|_{w,p} = \frac{\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma\left(\alpha+\frac{1}{p^*}\right)}, \quad \alpha > 0;$$

$$\|H(\alpha)\|_{w,p} = \frac{1}{\Gamma(\alpha)} \int_0^1 \theta^{-\frac{1}{p}} |\log\theta|^{\alpha-1} d\theta, \quad \alpha > 0;$$

$$\|\Gamma(\alpha)\|_{w,p} = \frac{\alpha p}{\alpha p - 1}, \quad \alpha p > 1;$$

$$\|E(\alpha)\|_{w,p} = \alpha^{-1/p}, \quad 0 < \alpha < 1.$$

Proof. It is elementary. □

The following corollary is an extension of Theorem 326 (p. 239 of [4]).

COROLLARY 2.2.

If x and w are non-negative sequences and w is decreasing, then

$$\sum_{n=1}^{\infty} w_n \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^p \leq p^{*p} \left(\sum_{n=1}^{\infty} w_n x_n^p \right).$$

Proof. For Cesaro operator we apply Corollary 2.1 with $\alpha = 1$. □

COROLLARY 2.3.

If $H(\mu)$ is the Hausdorff matrix operator on l_p and $p > 1$, then

$$\|H\|_p = \int_0^1 \theta^{-1/p} d\mu(\theta).$$

Proof. By taking $w_n = 1$ for all n, we have the corollary. □

COROLLARY 2.4.

Let $p > 1$. Then Cesaro, Hölder, Gamma and Euler operators map l_p into l_p. Also, we have

$$\|C(\alpha)\|_p = \frac{\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma\left(\alpha + \frac{1}{p^*}\right)}, \quad \alpha > 0;$$

$$\|H(\alpha)\|_p = \frac{1}{\Gamma(\alpha)} \int_0^1 \theta^{-\frac{1}{p}} |\log \theta|^{\alpha-1} d\theta, \quad \alpha > 0;$$

$$\|\Gamma(\alpha)\|_p = \frac{\alpha p}{\alpha p - 1}, \quad \alpha p > 1;$$

$$\|E(\alpha)\|_p = \alpha^{-1/p}, \quad 0 < \alpha < 1.$$

Proof. It is elementary. □

Theorem 2.2. *Let $p > 1$ and $H(\mu)$ be the Hausdorff matrix operator with condition (2). Then the Hausdorff matrix operator, $H(\mu)$, maps $d(w,p)$ into itself, and we have*

$$\|H\|_{d(w,p)} = \int_0^1 \theta^{-1/p} d\mu(\theta).$$

Proof. By Propositions 2.1 and 2.2, it is enough to consider non-negative decreasing sequences. For such sequences, we have $\|Hx\|_{d(w,p)} = \|Hx\|_{w,p}$, and so applying Theorem 1.1, we deduce the theorem. □

Example. Suppose $p > 1$. Since $\Gamma(1) = C(1)$ and they satisfy condition (2), we have

$$\|\Gamma(1)\|_{d(w,p)} = \|C(1)\|_{d(w,p)} = p^*.$$

Also

$$C(2) = \begin{bmatrix} 1 & 0 & & & \\ 2/3 & 1/3 & 0 & & \\ 3/6 & 2/6 & 1/6 & 0 & \\ 4/10 & 3/10 & 2/10 & 1/10 & 0 \\ . & . & . & . & . & . \end{bmatrix}$$

has condition (2) and so $\|C(2)\|_{d(w,p)} = p^*(2p)^*$.

3. Matrix operators on $d(w,1)$ and $e(w,\infty)$

Here we consider the upper bound problem for matrix operators from $d(v,1)$ into $d(w,1)$, and matrix operators from $e(w,\infty)$ into $e(v,\infty)$. If $x \in d(w,1)$, we denote norm of x with $\|x\|_{w,1}$ and if $x \in e(w,\infty)$, we denote norm of x with $\|x\|_{w,\infty}$. We write $\|A\|_{v,w,1}$ for the norm of A as an operator from $d(v,1)$ into $d(w,1)$, and $\|A\|_{w,v,\infty}$ for the norm of A as an operator from $e(w,\infty)$ into $e(v,\infty)$, and $\|A\|_{w,1}$ for the norm of A as an operator from $d(w,1)$ into itself, and $\|A\|_{w,\infty}$ for the norm of A as an operator from $e(w,\infty)$ into itself.

Theorem 3.1. *Suppose $A = (a_{i,j})$ is a matrix operator satisfying conditions* (1), (2) *and* (3). *If*

$$\sup \frac{S_n}{V_n} < \infty,$$

where $S_n = s_1 + \cdots + s_n$ and $s_n = \sum_{k=1}^{\infty} w_k a_{k,n}$ and $V_n = v_1 + \cdots + v_n$, then A is a bounded operator from $d(v,1)$ into $d(w,1)$, and also

$$\|A\|_{v,w,1} = \sup_n \frac{S_n}{V_n}.$$

Proof. By Proposition 2.1, it is sufficient to consider decreasing, non-negative sequences. Let x be in $d(v,1)$ such that $x_1 \geq x_2 \geq \cdots \geq 0$ and $M = \sup \frac{S_n}{V_n}$. Then

$$\|Ax\|_{w,1} = \sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} a_{n,k} x_k \right)$$

$$= \sum_{n=1}^{\infty} s_n x_n$$

$$= \sum_{n=1}^{\infty} S_n (x_n - x_{n+1})$$

$$\leq M \sum_{n=1}^{\infty} V_n (x_n - x_{n+1}).$$

Also, we have

$$\|x\|_{v,1} = \sum_{n=1}^{\infty} V_n (x_n - x_{n+1}).$$

Therefore

$$\|Ax\|_{w,1} \leq M\|x\|_{v,1},$$

and hence

$$\|A\|_{v,w,1} \leq M.$$

To show that the constant M is the best possible constant in the above inequality, we take $x_1 = x_2 = \cdots = x_n = 1$ and $x_k = 0$ for all $k \geq n+1$. Then

$$\|x\|_{v,1} = V_n, \quad \|Ax\|_{w,1} = S_n.$$

Therefore

$$\|A\|_{v,w,1} = M.$$

$$\square$$

In the following statement we obtain norm of general matrix operator from $e(w,\infty)$ into $e(v,\infty)$.

Theorem 3.2. *Suppose* $A = (a_{i,j})$ *is a matrix operator satisfying conditions* $(1),(2)$ *and* (3). *If*

$$\sup \frac{Z_n}{V_n} < \infty,$$

where $Z_n = z_1 + \cdots + z_n$ *and* $z_n = \sum_{k=1}^{\infty} w_k a_{n,k}$, *then* A *is a bounded operator from* $e(w,\infty)$ *into* $e(v,\infty)$, *and also*

$$\|A\|_{w,v,\infty} = \sup_n \frac{Z_n}{V_n}.$$

Proof. By Proposition 2.1, it is sufficient to consider decreasing, non-negative sequences. Let x be in $e(w,\infty)$ such that $x_1 \geq x_2 \geq \cdots \geq 0$ and $\|x\|_{w,\infty} = 1$. Then

$$X_n \leq W_n, \quad \forall n.$$

Let $y = Ax$ and $c_{n,j} = \sum_{i=1}^{n} a_{i,j}$. We have

$$
\begin{aligned}
Y_n = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{i,j} x_j \\
= \sum_{j=1}^{\infty} c_{n,j} x_j \\
= \sum_{j=1}^{\infty} (c_{n,j} - c_{n,j+1}) X_j \\
\leq \sum_{j=1}^{\infty} (c_{n,j} - c_{n,j+1}) W_j \\
= Z_n.
\end{aligned}
$$

If $C = \sup \frac{Z_n}{V_n}$, then

$$\sup \frac{Y_n}{V_n} \leq C,$$

and hence $\|A\|_{w,v,\infty} \leq C$.

Since $w \in e(w,\infty)$, $\|w\|_{w,\infty} = 1$ and $\|A(w)\|_{v,\infty} = C$, we have

$$\|A\|_{w,v,\infty} = C.$$

If A is a bounded matrix operator from $e(w,\infty)$ into $e(v,\infty)$, then A^t, the transpose matrix of A, is a bounded matrix operator of $d(v,1)$ into $d(w,1)$ and

$$\|A^t\|_{v,w,1} = \|A\|_{w,v,\infty}.$$

$\square$

Let (a_n) be a non-negative sequence with $a_1 > 0$, and $A_n = a_1 + \cdots + a_n$. The Nörlund matrix $N_a = (a_{n,k})$ is defined as follows:

$$a_{n,k} = \begin{cases} \frac{a_{n-k+1}}{A_n}, & 1 \leq k \leq n \\ 0, & k > n \end{cases}.$$

If $\alpha \geq 0$, the Cesaro matrix $C(\alpha)$ is matrix N_a with

$$a_n = \binom{n+\alpha-2}{n-1}.$$

The Copson matrix of order α is the transpose matrix of $C(\alpha)$, and we denote it with $C^t(\alpha)$. Also we denote $C = C(1)$ and $C^t = C^t(1)$.

In the following statements, we consider the norm of Cesaro and Copson matrices. It is enough to consider the sequence $\left(\frac{s_n}{v_n}\right)$ instead of $\left(\frac{S_n}{V_n}\right)$, because of the well-known fact listed in the following lemma.

Lemma 3.1. *If $m \leq \frac{s_n}{v_n} \leq M$ for all n, then $m \leq \frac{S_n}{V_n} \leq M$ for all n.*

Proof. It is elementary. $\square$

PROPOSITION 3.1.

If $w_n = \frac{1}{n}$ and $v_n = \frac{1}{n+\alpha}$ with $\alpha \geq 0$, then $C(2)$ is a bounded operator from $d(v,1)$ into $d(w,1)$ and also $C^t(2)$ is a bounded operator from $e(w,\infty)$ into $e(v,\infty)$, and

$$\|C(2)\|_{v,w,1} = \|C^t(2)\|_{w,v,\infty} = 2(\alpha+1).$$

Proof. We show that $\frac{s_n}{v_n} \leq \frac{s_1}{v_1}$ for all n. Therefore applying Lemma 3.1, we deduce that $\frac{S_n}{V_n} \leq \frac{S_1}{V_1} = s_1(\alpha+1)$, and by Theorem 3.1, we have

$$\|C(2)\|_{v,w,1} = 2(\alpha+1).$$

Since

$$s_1 = \sum_{k=1}^{\infty} \frac{1}{\frac{1}{2}k(k+1)} = 2,$$

for all n,

$$\frac{s_n}{v_n} = (n+\alpha) \sum_{k=n}^{\infty} \frac{1}{\frac{1}{2}k(k+1)} \frac{k-n+1}{k}$$

$$\leq 2(n+\alpha) \sum_{k=n}^{\infty} \frac{1}{k(k+1)}$$

$$\leq 2n \sum_{k=n}^{\infty} \frac{1}{k(k+1)} + 2\alpha \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

$$= 2 + 2\alpha = \frac{s_1}{v_1}.$$

This establishes the proof of the proposition.   □

PROPOSITION 3.2.

If $w_n = \frac{1}{n}$ and $v_n = \frac{1}{n^\alpha}$ with $0 \leq \alpha \leq 1$, then $C(2)$ is a bounded operator from $d(v,1)$ into $d(w,1)$ and also $C^t(2)$ is a bounded operator from $e(w,\infty)$ into $e(v,\infty)$, and

$$\|C(2)\|_{v,w,1} = \|C^t(2)\|_{w,v,\infty} = 2.$$

Proof. We show that $\frac{s_n}{v_n} \leq 2$ for all n. Therefore applying Lemma 3.1, we deduce that $\frac{S_n}{V_n} \leq 2$. For all n,

$$\frac{s_n}{v_n} = n^\alpha \sum_{k=n}^{\infty} \frac{1}{\frac{1}{2}k(k+1)} \frac{k-n+1}{k}$$

$$\leq 2n^\alpha \sum_{k=n}^{\infty} \frac{1}{k(k+1)}$$

$$\leq 2.$$

Since $\frac{s_1}{v_1} = 2$, we have $\sup \frac{S_n}{V_n} = 2$. This completes the proof of the proposition.   □

COROLLARY 3.1.

   If

$$\sup_{n} \frac{1}{V_n} \sum_{k=1}^{n} \frac{W_k}{k} < \infty,$$

then the Cesaro matrix C is a bounded operator from $e(w,\infty)$ into $e(v,\infty)$, and

$$\|C\|_{w,v,\infty} = \sup_{n} \frac{1}{V_n} \sum_{k=1}^{n} \frac{W_k}{k}.$$

Proof. By Theorem 3.1, we have

$$\|C^t\|_{v,w,1} = \sup \frac{S_n}{V_n}.$$

Since $s_n = \frac{W_n}{n}$ and $\|C^t\|_{v,w,1} = \|C\|_{w,v,\infty}$, we have the corollary.   □

Theorem 3.3. *Suppose*

$$r = \sup \frac{W_n}{n v_n} < \infty.$$

Then the Copson operator C^t maps $d(v,1)$ into $d(w,1)$ and

$$\|C^t\|_{v,w,1} \le r.$$

Proof. Since $s_n = \frac{W_n}{n}$, we have $\sup \frac{S_n}{v_n} \le r$. Hence

$$\|C^t\|_{v,w,1} = \sup \frac{S_n}{V_n} \le r.$$

$\square$

Theorem 3.4. *Suppose $v_n = \frac{1}{n^\alpha}$ and $W_n = n^{1-\alpha}$ with $0 \le \alpha \le 1$. Then the Copson operator C^t maps $d(v,1)$ into $d(w,1)$, and*

$$\|C^t\|_{v,w,1} = 1.$$

Therefore

$$\|C\|_{w,v,\infty} = 1.$$

Proof. For all n, $\frac{W_n}{n v_n} = 1$, and therefore $r = 1$. Hence

$$\|C^t\|_{v,w,1} \le 1.$$

Since $\frac{s_1}{v_1} = 1$, we deduce that

$$\|C^t\|_{v,w,1} = 1.$$

$\square$

Theorem 3.5. *Suppose $w_n = \frac{1}{n^\alpha}$ and $V_n = n^{1-\alpha}$ with $0 \le \alpha \le 1$. Then the Cesaro operator C maps $d(v,1)$ into $d(w,1)$, and*

$$\|C\|_{v,w,1} \le \frac{1}{1-\alpha} \zeta(1+\alpha).$$

Therefore the Copson operator C maps $e(w,\infty)$ into $e(v,\infty)$, and

$$\|C^t\|_{w,v,\infty} \le \frac{1}{1-\alpha} \zeta(1+\alpha).$$

Proof. By mean value theorem for all n, we have

$$\frac{1-\alpha}{n^\alpha} \le n^{1-\alpha} - (n-1)^{1-\alpha}.$$

Since $v_n = n^{1-\alpha} - (n-1)^{1-\alpha}$,

$$\frac{s_n}{v_n} \le \frac{n^\alpha}{1-\alpha} s_n,$$

and hence $\sup \frac{s_n}{v_n} \le \frac{1}{1-\alpha} \sup n^\alpha s_n$.

The sequence $(n^\alpha s_n)$ is decreasing (Lemma 2.7 of [6]), and therefore

$$\sup \frac{s_n}{v_n} \le \frac{1}{1-\alpha} s_1 = \frac{1}{1-\alpha} \zeta(1+\alpha).$$

This completes the proof of the theorem. $\square$

We recall that the Hilbert operator H is defined by the matrix

$$a_{i,j} = \frac{1}{i+j}.$$

Lemma 3.2. *If* $0 \leq \alpha \leq 1$, *then*

$$\sup_n n^\alpha \sum_{k=1}^\infty \frac{1}{k^\alpha(k+n)} = \frac{\pi}{\sin\alpha\pi}.$$

Proof. It is elementary.   □

In the following statement, we consider the upper bound of H.

Theorem 3.6. *Suppose* $w_n = \frac{1}{n^\alpha}$ *and* $V_n = n^{1-\alpha}$ *where* $0 \leq \alpha \leq 1$. *Then the Hilbert matrix operator* H *maps* $d(v,1)$ *into* $d(w,1)$, *and*

$$\|H\|_{v,w,1} \leq \frac{\pi}{(1-\alpha)\sin\alpha\pi}.$$

Therefore the Hilbert operator H *maps* $e(w,\infty)$ *into* $e(v,\infty)$, *and*

$$\|H\|_{w,v,\infty} \leq \frac{\pi}{(1-\alpha)\sin\alpha\pi}.$$

Proof. We have $\frac{s_n}{v_n} \leq \frac{n^\alpha}{1-\alpha}s_n$ which is similar to the previous theorem. Applying Lemma 3.2, we have $\sup_n n^\alpha s_n = \frac{\pi}{\sin\alpha\pi}$, and so

$$\frac{s_n}{v_n} \leq \frac{\pi}{(1-\alpha)\sin\alpha\pi}.$$

This completes the proof of the theorem.   □

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